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A multi-scaling analysis of the spin-up problem

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The response of a contained rotating fluid to a small, abrupt change in the rotation rate is analysed by multi-scaling methods. The procedure makes use of the fact that three different physical processes (inertial oscillations, spin-up response, diffusion) give rise to three different time scales. Since the flow is known to have a boundary-layer character, the variables are derived into interior and boundary-layer parts. The pertinent parameter separating the magnitudes of the amplitudes and the different time scales is the square root of the Ekman number $E^{\frac{1}{2}}$, so an expansion in powers of $E^{\frac{1}{2}}$ is used. The solution for a homogeneous fluid is derived first and is shown to be consistent with the solution of Greenspan & Howard (1963). The results are given in two forms: one is a direct deduction of the expansion method and is valid to $O(E)$; the other is obtained by regrouping the terms to derive a form apparently valid for indefinitely long times. When the fluid is stratified, the physical structure of the system is substantially more complicated, and so is the analysis. Exact results can be obtained for the case where the buoyancy N and the rotational Ω frequencies are the same. For the general case $F = N/\Omega \neq 1$, results valid for $t \gg 1$ can be obtained (where t is measured in units of Ω^{-1}). In both cases the exact lowest-order solution for the interior can be derived since it is independent of short time t. For the stratified fluid the elementary spin-up solution of Holton (1965) is part of the solution at $O(E^{\frac{1}{2}})$. The remaining part includes the long-time behaviour towards which the system tends as diffusive processes become dominant. The formulation of the long-time problem is complete at $O(E)$, but parts of it emerge from the analysis at lower order, and it is necessary to treat the lower-order system to obtain a consistent formulation at $O(E)$. In particular, it is possible to show that the thermal boundary condition, which does not affect the elementary spin-up solution, should be satisfied only by the long-time part of the problem. The complete, lowest-order response of the system includes a diffusive part which is quantitatively significant even for times of the order of one spin-up time. It is suggested here that the diffusive contribution may be responsible for parts of the discrepancy between elementary spin-up theory and recent experiments.

1. Introduction

The spin-up process, a term describing the response of a contained, uniformly rotating fluid to an abrupt change in the rotation rate of the container, has received considerable attention in the literature, because of its importance to transient geophysical flows (see Benton & Clark 1974 for an excellent review). The present paper outlines multi-timing analyses of the linear problem for the cases where the fluid is homogeneous and where it is stratified. Effects due to side-wall boundaries are omitted.

Using an effective combination of mathematical and physical reasoning, Greenspan & Howard (1963) solved the homogeneous problem by a direct application of Laplace transform theory. They presented the complete solution in the form of an inverse Laplace transform; they also exhibited the salient features of the flow with simple, approximate, analytical expressions.

One of the important results in the published solution is that the fluid responds to the change in rotation via processes that involve three widely separated time scales. Ekman layers are established in a time measured by the inverse of the rotation rate Ω^{-1}. The longest time scale is determined by diffusion; it is given by L^2/ν (L is the half-depth of the container and ν is the kinematic viscosity). This long time scale can also be written in terms of the Ekman number

$$E = \nu/\Omega L^2 \quad \text{as} \quad E^{-1}\,\Omega^{-1}.$$

The intermediate time scale is the logarithmic mean of the above two and is $E^{-\frac{1}{2}}\Omega^{-1}$. The fluid processes that are associated with this intermediate time are those that effectively contain the spin-up process. As Greenspan & Howard have shown, diffusion serves only to dissipate the residual, weak, inertial oscillations.

Once one recognizes the existence of separated time scales, a multi-timing treatment of the problem is strongly suggested. We have carried out such an analysis and report the results for the homogeneous problem in §§3 and 4. The present analysis yields no new information about the physics, but it sheds a good deal of light on the method of approximating the exact solution by means of a multi-scaling procedure. In particular, zero-, first- and second-order approximations to the interior azimuthal velocity are obtained. The results indicate that the expansion in powers of $E^{\frac{1}{2}}$ is not uniformly valid in time. But, by regrouping and rewriting the results, we obtain a higher-order form of the solution of Greenspan & Howard that is uniformly valid in time and agrees with the multi-scaling expansion up to $O(E)$. This procedure for the homogeneous problem is important, because it clarifies some of the interpretive difficulties of the multi-timing expansion and makes it easier to understand the results for the stratified case.

The real power of the multi-scaling approach is felt in the analysis of the spin-up of a stratified fluid. This problem cannot be tackled directly, the order of the full system of equations being much too high; and it is necessary to make use of the simpler subsystems of the multi-scaling method.

The stratified problem, treated in §§5 and 6, differs substantially from the homogeneous problem in several ways. The presence of stratification introduces physical processes and mechanisms wholly lacking in the homogeneous system. In particular, stratification inhibits the penetration of the initial spin-up process to a boundary layer whose thickness is determined by F, the ratio of the Brunt–Väisälä frequency $[(-g/\rho)(\partial\rho/\partial z)]^{\frac{1}{2}}$ to twice the rotation frequency 2Ω. Earlier work on stratified spin-up (Holton 1965; Sakurai 1969; Walin 1969; Buzyna &

Veronis 1971) concentrated on this part of the problem. The multi-scaling analysis carried through the initial spin-up time gives results in agreement with the earlier ones. In the special cases where the complete problem is tractable, the present spin-up solutions also include higher frequency inertio-gravity waves associated with the short-time scales.

Another distinctive feature of the stratified problem in contrast to the homogeneous case is that the final steady state is structurally different from that given by the initial spin-up analysis. The function containing this feature emerges from that part of the multi-scaling solution that includes the initial spin-up time but it appears as a completely undetermined function of space and (long) time at that order. Its structure is resolved only when the long-time problem is solved.

An associated, interesting aspect of the multi-scaling approach is that the boundary conditions for the final steady state are determined by the requirement that successive terms of the expansion be of decreasing magnitude. These boundary conditions are necessary to obtain the final solution, and would normally be formulated as such for the steady problem. However, the order at which they appear in the multi-scaling analysis is critical. If they appear at too low an order, they lead to a contradiction.† If they appear at too high an order, an under-determined problem occurs. It is especially interesting to find that the proper boundary conditions emerge unambiguously only when the shortest time scale is included in the analysis.

Although our treatment of the homogeneous problem is for the cylindrical geometry of the laboratory experiment, the analysis for the stratified problem is for a single harmonic in one horizontal direction. Because of the complication introduced by the stratification, we chose to adopt this simplification in the horizontal structure, so that we could concentrate on the important physics. Even so, the complete problem is not fully tractable, and we had to settle for approximate results when F is arbitrary. For the case $F = 1$, the full lowest-order solution can be derived, and we have used the results for that case as a check on the approximate approach required for $F \neq 1$.

Throughout this paper we avoid treating actual lateral boundaries. The presence of lateral boundaries can affect the homogeneous system, by imposing a constraint on the lateral structure of inertial oscillations in the interior, a process that presumably can be treated by incorporating appropriate Stewartson layers at the side walls in the multi-scaling procedure. But the analysis becomes considerably more complicated, and we have not carried it through. For the stratified problem it is known (Sakurai 1969) that different lateral boundary conditions can alter the elementary spin-up solution significantly. Here again, the complications for the full problem are non-trivial.

† Those who are familiar with the solutions of Holton (1965), Walin (1969) and Sakurai (1969) may recall that no condition on temperature can be specified at the upper and lower boundaries. Their solutions satisfy boundary conditions associated only with the velocities.

2. Formulation of homogeneous spin-up problem

The equations for a viscous incompressible fluid in a co-ordinate system rotating about the z axis with constant angular velocity Ω are

$$\partial \mathbf{v}_*/\partial t_* + \mathbf{v}_*.\nabla\mathbf{v}_* + 2\Omega\mathbf{k}\times\mathbf{v}_* = -\nabla[(P_*/\rho_*) - \tfrac{1}{2}\Omega^2(x_*^2+y_*^2)] + \nu\nabla^2\mathbf{v}_*,$$
$$\nabla.\mathbf{v}_* = 0.$$

The auxiliary conditions that describe the spin-up problem between two disks are: $\mathbf{v}_* = 0$ for $t_* \leqslant 0$ and $\mathbf{v}_* = \Delta\Omega\mathbf{k}\times\mathbf{r}_*$ at $z_* = \pm L$ and for $t_* > 0$. The following non-dimensional variables are introduced:

$$\mathbf{r}_* = L\mathbf{r}, \quad t_* = \Omega^{-1}t, \quad \mathbf{v}_* = \epsilon L\Omega\mathbf{v}, \quad \epsilon L^2\Omega^2 p = (P_*/\rho_*) - \tfrac{1}{2}\Omega^2(x_*^2+y_*^2).$$

$\epsilon = \Delta\Omega/\Omega$ is the Rossby number and the unstarred variables are dimensionless. Then the equations become

$$\partial\mathbf{v}/\partial t + \epsilon\mathbf{v}.\nabla\mathbf{v} + 2\mathbf{k}\times\mathbf{v} = -\nabla p + E\nabla^2\mathbf{v} \quad (\nabla.\mathbf{v} = 0). \qquad (2.1)$$

$E = \nu/\Omega L^2$ is the Ekman number. The initial and boundary conditions are

$$\left.\begin{aligned} \mathbf{v} = 0 \quad \text{for} \quad t \leqslant 0, \\ \mathbf{v} = \mathbf{k}\times\mathbf{r} \quad \text{at} \quad z = \pm 1 \quad \text{for} \quad t > 0. \end{aligned}\right\} \qquad (2.2)$$

We consider the linear, axisymmetric problem (i.e. $\epsilon = 0$ and $\partial/\partial\theta = 0$). Furthermore, we can make use of the simple spatial dependence of the boundary conditions to note that the r-dependence is eliminated if we choose

$$u = U(z,t)\,r, \quad v = V(z,t)\,r, \quad w = W(z,t), \quad p = P(t)\,r^2 + \Pi(z,t).$$

Observe that the pressure field is divided into a part dependent on, and a part independent of, r. With subscripts denoting partial derivatives, the equations take the component form

$$U_t - 2V = -2P + EU_{zz}, \quad V_t + 2U = EV_{zz}, \quad W_t = -\Pi_z + EW_{zz}, \quad 2U + W_z = 0. \qquad (2.3)$$

The initial and boundary conditions are

$$\left.\begin{aligned} U = V = W = 0 \quad \text{for} \quad t \leqslant 0, \\ U = W = 0, \quad V = 1 \quad \text{at} \quad z = \pm 1 \quad \text{for} \quad t > 0. \end{aligned}\right\} \qquad (2.4)$$

Time scales

We may expect the time dependence to comply with three different physical processes. (i) Inertial motions have a dimensional scale of Ω^{-1}. In dimensionless form the time scale is $O(1)$. (ii) Ekman layers and associated Ekman pumping are established within a time scale of order 1. The fluid in the interior will be affected by the vertical velocity induced by Ekman pumping or suction into the boundary layer. In dimensionless units this induced vertical velocity is $O(E^{\frac{1}{2}})$. Since a particle of fluid near mid-depth is unit distance from the upper and lower boundaries, it will feel the effects of the boundaries in a (non-dimensional) time of order $E^{-\frac{1}{2}}$. This is the time required for a vertical traverse over unit distance by a particle moving with velocity $E^{\frac{1}{2}}$. (iii) Diffusion will transmit effects of the boundaries to the interior in a time scale $O(E^{-1})$.

Hence, we may expect the time dependence to reflect the three time scales. We shall therefore write any variable χ as a function of these three times. Thus, if

$$t = t, \quad \tau = E^{\frac{1}{2}}t, \quad T = Et, \tag{2.5}$$

we write any variable χ as $\chi(t, \tau, T, z)$, so that

$$\frac{\partial \chi}{\partial t} = \frac{\partial \chi}{\partial t} + E^{\frac{1}{2}}\frac{\partial \chi}{\partial \tau} + E\frac{\partial \chi}{\partial T}. \tag{2.6}$$

The equations (2.3) thus take the form

$$\left.\begin{aligned} U_t + E^{\frac{1}{2}}U_\tau + EU_T - 2V &= -2P + EU_{zz}, \\ V_t + E^{\frac{1}{2}}V_\tau + EV_T + 2U &= EV_{zz}, \\ W_t + E^{\frac{1}{2}}W_\tau + EW_T &= -\Pi_z + EW_{zz}. \end{aligned}\right\} \tag{2.7}$$

Interior and boundary-layer variables

Experience with rotating fluids indicates that, for linear problems of this type, the variables are composed of a part that has vertical derivatives of order 1 throughout the fluid plus a part that has vertical derivatives of order $E^{-\frac{1}{2}}$ near the upper and lower boundaries, and decays to zero in a distance of order $E^{\frac{1}{2}}$ from these boundaries. We denote the interior part by subscript I and the boundary-layer part by subscript b. Furthermore, for vertical derivatives in the boundary layer we write

$$\partial/\partial z = \mp E^{-\frac{1}{2}}(\partial/\partial \zeta) \quad \text{near} \quad z = \pm 1; \tag{2.8}$$

$\partial/\partial \zeta$ is $O(1)$, and ζ is defined by

$$\zeta = (1 \mp z)E^{-\frac{1}{2}} \quad \text{near} \quad z = \pm 1. \tag{2.9}$$

Ordering of variables and equations

Since time scales and spatial derivatives have magnitudes related to integral powers of $E^{\frac{1}{2}}$, we shall expand all variables and equations in powers of $E^{\frac{1}{2}}$. Hence, we write for any variable

$$\chi = \chi_I + \chi_b, \tag{2.10}$$

and

$$\chi_I = \sum_0^\infty E^{n/2}\chi_n, \quad \chi_b = \sum_0^\infty E^{n/2}\tilde{\chi}_n. \tag{2.11}$$

The formal problem can now be reduced to a hierarchy of problems. For the interior variables we have

$$U_{nt} + U_{(n-1)\tau} + U_{(n-2)T} - 2V_n = -2P_n + U_{(n-2)zz}, \tag{2.12a}$$

$$V_{nt} + V_{(n-1)\tau} + V_{(n-2)T} + 2U_n = V_{(n-2)zz}, \tag{2.12b}$$

$$W_{nt} + W_{(n-1)\tau} + W_{(n-2)T} = -\Pi_{nz} + W_{(n-2)zz}, \tag{2.12c}$$

$$2U_n + W_{nz} = 0. \tag{2.12d}$$

For the boundary-layer variables near $z = -1$ the equations take the form

$$\tilde{U}_{nt} + \tilde{U}_{(n-1)\tau} + \tilde{U}_{(n-2)T} - 2\tilde{V}_n = \tilde{U}_{n\zeta\zeta}, \tag{2.13a}$$

$$\tilde{V}_{nt} + \tilde{V}_{(n-1)\tau} + \tilde{V}_{(n-2)T} + 2\tilde{U}_n = \tilde{V}_{n\zeta\zeta}, \tag{2.13b}$$

$$\tilde{W}_{(n-1)t} + \tilde{W}_{(n-2)\tau} + \tilde{W}_{(n-3)T} = -\tilde{\Pi}_{n\zeta} + \tilde{W}_{(n-1)\zeta\zeta}, \tag{2.13c}$$

$$2\tilde{U}_{n-1} + \tilde{W}_{n\zeta} = 0, \tag{2.13d}$$

where, since $\tilde{P}_n$ is independent of ζ and must decay as $\zeta \to \infty$, it must vanish identically.

These equations are valid for $n = 0, 1, 2, \ldots$. Wherever a negative subscript is encountered the variable is to be replaced by zero. Instead of writing the boundary-layer system near $z = 1$, we make use of the obvious vertical symmetry of the problem to impose boundary conditions on the interior variables at $z = 0$. Thus,

$$U_{nz} = V_{nz} = W_n = 0 \quad \text{at} \quad z = 0. \tag{2.14}$$

Hence, the domain of interest is $0 \geqslant z \geqslant -1$.

The boundary conditions (2.4) at $z = -1$ now become

$$U_n^0 + \tilde{U}_n^0 = 0, \quad W_n^0 + \tilde{W}_n^0 = 0, \quad V_0^0 + \tilde{V}_0^0 = 1, \quad V_n^0 + \tilde{V}_n^0 = 0 \quad \text{for} \quad n > 0, \tag{2.15}$$

where superscript 0 denotes the boundary value at $z = -1$. All variables vanish for $t \leqslant 0$.

3. Solutions for homogeneous fluid problem

Zero-order system

Equations (2.13c, d) for $n = 0$ yield $\tilde{\Pi}_{0\zeta} = 0$, $\tilde{W}_{0\zeta} = 0$, so that $\tilde{\Pi}_0$ and $\tilde{W}_0$ vanish, since they must decay as $\zeta \to \infty$. Thus, from (2.14) and (2.15), we note that $\tilde{W}_0$ must vanish at $z = 0, -1$. Hence, W_0 can be a solution of the zero-order system only if it has the form

$$W_0 = \sum_1^\infty \sin n\pi z \,[A_n(\tau, T) \sin 2t + B_n(\tau, T) \cos 2t] \tag{3.1}$$

with $A_n(0,0) = B_n(0,0) = 0$. These are free solutions. A form similar to (3.1) appears for each W_n. Rather than carry these forms and attempt to treat them separately, we observe that, from the original dissipative system (2.3) and (2.4), we can show that the mean-square amplitude of any disturbance with zero initial and boundary values must vanish identically. Hence $A_n \equiv B_n \equiv 0$ and $W_0 = 0$. Furthermore, from (2.12) with $n = 0$, it is easy to verify that

$$U_0 = 0, \quad W_0 = 0, \quad V_{0t} = 0, \quad V_{0z} = 0. \tag{3.2}$$

Therefore, the zero-order results are consistent with the Taylor–Proudman theorem and, in particular,

$$V_0 = V_0(\tau, T). \tag{3.3}$$

Equations (2.13a, b) with $n = 0$ yield the Ekman-layer equations

$$\tilde{U}_{0t} - 2\tilde{V}_0 = \tilde{U}_{0\zeta\zeta}, \quad \tilde{V}_{0t} + 2\tilde{U}_0 = \tilde{V}_{0\zeta\zeta}, \tag{3.4}$$

with boundary conditions from (2.15)

$$\tilde{U}_0^0 = 0, \quad \tilde{V}_0^0 = 1 - V_0^0 = 1 - V_0, \tag{3.5}$$

where we have used the results (3.2) and (3.3) in (3.5). The solution to this system is most easily derived by writing $\tilde{\phi}_0 = \tilde{U}_0 + i\tilde{V}_0$, so that (3.4) becomes

$$\tilde{\phi}_{0t} + 2i\tilde{\phi}_0 = \tilde{\phi}_{0\zeta\zeta}. \tag{3.6}$$

The solution to this problem is straightforward. It is

$$\tilde{U}_0 = (1-V_0)\int_0^t \frac{\zeta \exp(-\zeta^2/4\alpha)}{2\alpha(\pi\alpha)^{\frac{1}{2}}} \sin 2\alpha \, d\alpha, \tag{3.7a}$$

$$\tilde{V}_0 = (1-V_0)\int_0^t \frac{\zeta \exp(-\zeta^2/4\alpha)}{2\alpha(\pi\alpha)^{\frac{1}{2}}} \cos 2\alpha \, d\alpha. \tag{3.7b}$$

Since (2.13*d*) with $n = 1$ yields

$$\tilde{W}_1 = 2\int_\zeta^\infty \tilde{U}_0 \, d\zeta,$$

we obtain

$$\tilde{W}_1 = 2(1-V_0)\int_0^t \frac{\exp(-\zeta^2/4\alpha)}{(\pi\alpha)^{\frac{1}{2}}} \sin 2\alpha \, d\alpha. \tag{3.8}$$

First-order system

From (2.12) with $n = 1$ we obtain

$$\left.\begin{aligned} U_{1zt} - 2V_{1z} = 0, \quad V_{1t} + 2U_1 = -V_{0\tau},\\ 2U_1 + W_{1z} = 0, \quad W_{1zztt} + 4W_{1zz} = 0. \end{aligned}\right\} \tag{3.9}$$

The solution for W_1 must satisfy the condition (2.15):

$$W_1^0 = -\tilde{W}_1^0 = -2(1-V_0)\,S(2t), \tag{3.10}$$

where $S(2t)$ is the Fresnel integral defined by

$$S(2t) = \int_0^t (\sin 2\alpha/(\pi\alpha)^{\frac{1}{2}})\, d\alpha.$$

Since $\tilde{W}_1^0 \to (1-V_0)$ as $t \to \infty$, solutions of W_1 proportional to $\exp(2it)$ must be free solutions of the form (3.1). These must vanish identically by the argument given earlier. Hence, W_{1zz} must vanish and W_1 can at most be a linear function of z, which satisfies (3.10) at $z = -1$ and vanishes at $z = 0$. Therefore,

$$W_1 = 2z(1-V_0)\,S(2t), \quad U_1 = -(1-V_0)\,S(2t). \tag{3.11}$$

Furthermore, from (3.9) V_1 must be independent of z, and we obtain

$$V_{1t} = -V_{0\tau} + 2(1-V_0)\,S(2t). \tag{3.12}$$

In order that our expansion $V = V_0 + E^{\frac{1}{2}}V_1 + \ldots$ remain uniformly valid, we require that V_1 be bounded as $t \to \infty$. We shall impose this condition (sometimes called the non-secularity condition) by requiring simply that $V_{1t} \to 0$ as $t \to \infty$. Then (3.12) yields

$$V_{0\tau} + V_0 = 1 \tag{3.13}$$

or

$$V_0 = 1 - f(T)\exp(-\tau) \quad (f(0) = 1). \tag{3.14}$$

Hence, the equation for V_1 provides a partial solution for V_0. At this level the first-order variables are

$$U_1 = -f(T)\exp(-\tau)S(2t), \tag{3.15a}$$

$$V_1 = f(T)\exp(-\tau)\left[2\int_0^t S(2\alpha)\,d\alpha - t\right], \tag{3.15b}$$

$$W_1 = 2zf(T)\exp(-\tau)S(2t). \tag{3.15c}$$

The arbitrary function of τ and T arising from the integration of (3.12) is part of a free solution which, when taken with its corresponding boundary-layer part, can be shown to vanish because of the argument given earlier for free solutions of W_n.

Boundary layer at first order

With $n = 1$ (2.13a, b) yield

$$\tilde{U}_{1t} - 2\tilde{V}_1 = \tilde{U}_{1\zeta\zeta} - \tilde{U}_{0\tau}, \quad \tilde{V}_{1t} + 2\tilde{U}_1 = \tilde{V}_{1\zeta\zeta} - \tilde{V}_{0\tau}; \tag{3.16}$$

or, in terms of $\tilde{\phi}_1 (= \tilde{U}_1 + i\tilde{V}_1)$,

$$\tilde{\phi}_{1t} + 2i\tilde{\phi}_1 = \tilde{\phi}_{1\zeta\zeta} - \tilde{\phi}_{0\tau}. \tag{3.17}$$

The boundary conditions become

$$\tilde{\phi}_1^0 = -\phi_1^0 = -\phi_1. \tag{3.18}$$

The last equality is a consequence of the fact that U_1 and V_1 are independent of z.

The solution to the $\tilde{\phi}_1$ problem is considerably messier than the $\tilde{\phi}_0$ problem. Since our primary goal is to obtain the boundary condition necessary for W_2, we can take the Laplace transform of $\tilde{\phi}_1$ in time and postpone the inversion until it is required. Writing $L(\tilde{\phi}_1) = \hat{\tilde{\phi}}$, we obtain

$$\hat{\tilde{\phi}}_1 = \left[\hat{\tilde{\phi}}_1^0 - \frac{\tilde{\phi}_{0\tau}^0\,\zeta}{2s(s+2i)^{\frac{1}{2}}}\right]\exp\left(-(s+2i)^{\frac{1}{2}}\zeta\right). \tag{3.19}$$

s is the transform variable. Furthermore, from (3.9) and (3.18) we have

$$\hat{\tilde{\phi}}_1^0 = -\hat{\phi}_1 = \frac{s-2i}{2s}\hat{W}_{1z} + \frac{i}{s^2}V_{0\tau}. \tag{3.20}$$

$\hat{W}_{1z}$ is derived from the Laplace transform of (3.15c); it is

$$\hat{W}_{1z} = \frac{-if(T)}{s}\exp(-\tau)\left[(s-2i)^{-\frac{1}{2}} - (s+2i)^{-\frac{1}{2}}\right]. \tag{3.21}$$

A Laplace transform, then integration of (2.13d) from $\zeta = 0$ to $\zeta = \infty$, yield

$$\hat{\tilde{W}}_2^0 = -2\,\mathrm{Re}\int_0^\infty \hat{\tilde{\phi}}_1 d\zeta = -\,\mathrm{Re}\left\{\frac{\hat{W}_{1z}(s-2i)}{s(s+2i)^{\frac{1}{2}}} + \frac{2iV_{0\tau}}{s\,(s+2i)^{\frac{1}{2}}}\left(\frac{1}{s} + \frac{1}{2(s+2i)}\right)\right\}; \tag{3.22}$$

and the inverse Laplace transform gives

$$\tilde{W}_2^0 = f(T)\exp(-\tau)\left\{2tS(2t) - 2\int_0^t (t-\alpha)\,J_0(2\alpha)\,d\alpha + 1 - \cos 2t\right\}. \tag{3.23}$$

J_0 is the zero-order Bessel function.

Second-order system

Setting $n = 2$ in (2.12) yields

$$\left.\begin{aligned} &U_{2zt} - 2V_{2z} = 0, \quad V_{2t} + 2U_2 = -V_{0T} - V_{1\tau}, \\ &2U_2 + W_{2z} = 0, \quad W_{2zztt} + 4W_{2zz} = 0. \end{aligned}\right\} \tag{3.24}$$

Once again we can conclude that W_2 is a linear function of z. The boundary value at $z = -1$ is given by $-\tilde{W}_2$ at $\zeta = 0$, so that $W_2 = z\tilde{W}_2^0$. Hence, we can substitute known values for each term with subscript 0 or 1 in the equation for V_{2t} in (3.24), to obtain

$$V_{2t} = f(T)\exp(-\tau)\left\{2\int_0^t S(2\alpha)\,d\alpha - t + 2tS(2t) - 2\int_0^t (t-\alpha)\,J_0(2\alpha)\,d\alpha + 1 - \cos 2t\right\} + f_T\exp(-\tau). \tag{3.25}$$

We shall eliminate the most serious secular terms from V_2 by requiring†

$$\lim_{t\to\infty} t^{-1}\int_0^t V_{2t}\,dt = 0. \tag{3.26}$$

The only terms that do not vanish naturally over the infinite time interval in (3.25) are those that have a finite value asymptotically as $t \to \infty$. Thus, taking the asymptotic limit of V_{2t} in (3.25), we find

$$V_{2t} \to \exp(-\tau)\left[f_T + \tfrac{3}{4}f\right] - \exp(-\tau) f\left[(t/\pi)^{\frac{1}{2}} + 1\right]\cos 2t. \tag{3.27}$$

When (3.27) is substituted into (3.26), we find that the terms involved in the second bracket vanish because of the averaging. The first term does not; and our requirement for a uniformly valid solution means that we must take

$$f_T + \tfrac{3}{4}f = 0. \tag{3.28}$$

Since $f(0) = 1$, the solution of (3.28) is

$$f = \exp(-\tfrac{3}{4}T). \tag{3.29}$$

Hence, our solutions for V_0, V_1 and V_2 are

$$V_0 = 1 - \exp(-\tau - \tfrac{3}{4}T), \tag{3.30a}$$

$$V_1 = \exp(-\tau - \tfrac{3}{4}T)\left[2\int_0^t S(2\alpha)\,d\alpha - t\right], \tag{3.30b}$$

$$V_2 = \exp(-\tau - \tfrac{3}{4}T)\left\{2t\int_0^t (t-\alpha)\left[\sin 2\alpha/(\pi\alpha)^{\frac{1}{2}}\right]d\alpha - \int_0^t (t-\alpha)^2 J_0(2\alpha)\,d\alpha - \tfrac{1}{2}t^2 + \tfrac{1}{4}t - \tfrac{1}{4}\sin 2t\right\}. \tag{3.30c}$$

The total interior swirl velocity to $O(E)$ is

$$V_I = V_0 + E^{\frac{1}{2}}V_1 + EV_2. \tag{3.31}$$

† Condition (3.26) is somewhat weaker than the requirement that $V_{2t} \to 0$. There can be growing oscillations, as long as they do not contribute to the mean time behaviour.

4. Discussion of homogeneous solution

It is instructive to rearrange the terms in (3.31) when the values of V_0, V_1 and V_2 are substituted. We obtain

$$V_I = 1-\exp(-\tau-\tfrac{3}{4}T)\,[1+E^{\frac{1}{2}}t+\tfrac{1}{2}Et^2+\tfrac{3}{4}Et]$$
$$+2\exp(-\tau-\tfrac{3}{4}T)\int_0^t[\sin 2\alpha/(\pi\alpha)^{\frac{1}{2}}]\,[1+E^{\frac{1}{2}}t+\tfrac{1}{2}Et^2-1-E^{\frac{1}{2}}\alpha-\tfrac{1}{2}E\alpha^2]\,d\alpha$$
$$+E\exp(-\tau-\tfrac{3}{4}T)\int_0^t(t-\alpha)^2[[\sin 2\alpha/(\pi\alpha)^{\frac{1}{2}}]+2\cos 2\alpha-J_0(2\alpha)]\,d\alpha. \quad (4.1)$$

Now, to the orders considered here, we have

$$\left.\begin{aligned} 1+E^{\frac{1}{2}}t+\tfrac{1}{2}Et^2+\tfrac{3}{4}Et &= \exp(\tau+\tfrac{3}{4}T),\\ 1+E^{\frac{1}{2}}t+\tfrac{1}{2}Et^2 &= \exp\tau,\\ 1+E^{\frac{1}{2}}\alpha+\tfrac{1}{2}E\alpha^2 &= \exp\alpha; \end{aligned}\right\} \quad (4.2)$$

and (4.1) can be rewritten as

$$V_I = \exp(-\tfrac{3}{4}T)\left[2S(2t)-2\exp(-\tau)\int_0^t(\sin 2\alpha/(\pi\alpha)^{\frac{1}{2}})\exp(E^{\frac{1}{2}}\alpha)\,d\alpha\right]$$
$$+E\exp(-\tau-\tfrac{3}{4}T)\int_0^t(t-\alpha)^2[\sin 2\alpha/(\pi\alpha)^{\frac{1}{2}}+2\cos 2\alpha-J_0(2\alpha)]\,d\alpha. \quad (4.3)$$

For large times, the term in (4.3) with coefficient E tends to

$$E\exp(-\tau-\tfrac{3}{4}T)\,[\tfrac{3}{4}t+\tfrac{1}{32}-\tfrac{1}{2}\sin 2t]. \quad (4.4)$$

Of the terms in (4.4), the first is obviously the most important. Since $tE = T$, we have $\frac{3}{4}tE = \frac{3}{4}T$. The solutions given above, as well as the homogeneous solution, suggest that $\frac{3}{4}T$ be written as $\exp(\frac{3}{4}T)-1$ (correct to $O(E)$), so that

$$\exp(-\tau-\tfrac{3}{4}T)(\tfrac{3}{4}T) = \exp(-\tau-\tfrac{3}{4}T)(1+\tfrac{3}{4}T-1)$$
$$= \exp(-\tau-\tfrac{3}{4}T)(\exp(\tfrac{3}{4}T)-1). \quad (4.5)$$

Hence, to $O(E)$, (4.3) can be rewritten as

$$V_I = \exp(-\tfrac{3}{4}T)\left[2S(2t)-2\exp(-\tau)\int_0^t\sin 2\alpha/(\pi\alpha)^{\frac{1}{2}}\exp(E^{\frac{1}{2}}\alpha)\,d\alpha\right]$$
$$+\exp(-\tau)(1-\exp(-\tfrac{3}{4}T)), \quad (4.6)$$

where we have neglected terms whose amplitude is always less than $E\exp(-\tau)$.

Equation (3.18) of Greenspan & Howard can be rewritten as

$$V_I = 2S(2t)-2\exp(-\tau)\int_0^t\sin 2\alpha/(\pi\alpha)^{\frac{1}{2}}\exp(E^{\frac{1}{2}}\alpha)\,d\alpha. \quad (4.7)$$

Hence, our result provides a long-time correction to theirs. Since (4.7) contains the essential physics, the correction in (4.6) is of no interest physically; but it does add the proper detail as $t\to E^{-1}$.

We make two observations about the solutions obtained. (i) For $t \ll E^{-\frac{1}{2}}$

the amplitude of V_1 is $O(E^{\frac{1}{2}})$. This can be seen by expanding the exponentials in (4.7) in Taylor series, keeping only the linear terms, so that, for $\tau < 1$,

$$V_I = 2E^{\frac{1}{2}} \int_0^t [(t-\alpha)\sin 2\alpha/(\pi\alpha)^{\frac{1}{2}}]\, d\alpha. \tag{4.8}$$

Thus, V_I starts off as $O(E^{\frac{1}{2}})$, and becomes $O(1)$ only as $t \to E^{-\frac{1}{2}}$. An even simpler form can be obtained for $0 \ll t \ll E^{-\frac{1}{2}}$, when the asymptotic limit of V_I is given by

$$V_I \to E^{\frac{1}{2}}[t - \tfrac{1}{4} - \tfrac{1}{4}(\sin 2t/(\pi t)^{\frac{1}{2}})]. \tag{4.9}$$

The growth of V_I to $O(1)$ in a time of $O(E^{-\frac{1}{2}})$ is clear even though the approximation is not really valid at $t = E^{-\frac{1}{2}}$. We also see that the amplitude of the inertial oscillations is $O(E^{\frac{1}{2}})$. Hence, initially the oscillations have the same amplitude as the spin-up response; but as $t \to E^{-\frac{1}{2}}$ they are relatively less important. (ii) Our formal expansion appears not to be uniformly valid, because V_2 contains a term that goes asymptotically like $t^{\frac{1}{2}} \sin 2t \exp(-\tau - \frac{3}{4}T)$. Hence, for $t \approx E^{-1}$ we have $EV_2 > E^{\frac{1}{2}}V_1$. However, when we regroup the different terms for V_I, as we did in (4.1), and rewrite them as in (4.2), we obtain the result (4.3), which has no apparent non-uniform behaviour for large times. Although the regrouping (4.1) and the approximations (4.2) are not unique, our procedure is strongly supported by the appearance of the same types of exponentials in the other terms. The fact that our final result agrees with the Greenspan & Howard solution is an added indication that the procedure is correct.

As for the expansion procedure itself, we observe that the structure that emerges from the analysis is a step-by-step approximation to the correct solution. The apparent non-uniform character of the expansion is contained also in the approximate representation of the Greenspan & Howard solution, if one tries to use their approximation beyond the indicated time. Thus, to $O(E)$, the solution (3.31) together with (3.30) is as good as those given by (4.6) or (4.7). The closed forms of the latter look better; but these are also valid only for times up to $O(E^{-1})$. It is possible, of course, that (4.6) is valid for much longer times, but there is no way of knowing this without including longer times ($E^{-\frac{3}{2}}, E^{-2}$, etc.) in the expansion and carrying out the analysis.

The multi-timing analysis for the damped harmonic oscillator (Nayfeh 1973) exhibits many of the features obtained above, but in a much simpler context where the exact solution is known. The obvious choice for lowest-order time is the frequency of the undamped oscillator. In the exact solution the real frequency is modified by the damping, and it is this modification that emerges with the higher-order corrections of the multi-timing method. But in this case, too, the first-order results contain the essential physics.

Another aspect of the spin-up solution is that we cannot be sure that the multi-scale procedure works with a problem that has an expanding space scale. We know that the prototype transient Ekman layer (with a spatially uniform stress applied at the surface from time $t = 0$) yields a response whose effects penetrate infinitely far into the fluid for sufficiently long time. In particular, the horizontal velocity decays as $\zeta \exp(-\zeta^2/4t)$. In the present problem, if we integrate this expression from $\zeta = 0$ to $\zeta = E^{-\frac{1}{2}}$ (the latter limit being as close to ∞ as ζ, in fact, approaches), we obtain the results derived above for $\tilde{W}_2^0$. However, for $t \sim E^{-1}$ and $\zeta \sim E^{-\frac{1}{2}}$, the argument of the exponential is $O(1)$. Hence, the evaluation should not really be made at infinity, since ζ never exceeds $E^{-\frac{1}{2}}$ and t does extend formally to E^{-1}. This point is raised again in the stratified problem, where we present an argument to eliminate penetration of thermal effects to the interior via boundary-layer processes. In the homogeneous case, boundary-layer penetration does *not seem to occur.*

An additional consideration is that the dependence of the lowest-order variables on the τ and T time scales is obtained from the analysis in the asymptotic range $t \rightarrow \infty$. This procedure is standard for multi-timing analysis. It means that the long-time behaviour for the lowest-order quantities can be obtained, even when it is not possible to derive the detailed, short-time solution for higher-order variables. In the analysis for the general stratified problem we have to resort to the asymptotic analysis because we cannot obtain the complete solution.

Finally, we point out that our results for the functional form of the τ and T time scales can be obtained by neglecting the smallest time scale t from the outset. This simpler procedure yields solutions that are proportional to

$$\exp(-\tau - \tfrac{3}{4}T),$$

just as the more complete analysis does. From the analysis of the stratified problem, it appears that the simpler procedure works for the homogeneous case, because the spatial form of the elementary spin-up solution coincides with the final, steady-state spatial structure. One cannot proceed in the same formal way and obtain the correct τ and T time behaviour for the stratified problem. We shall return to this point later.

5. The stratified spin-up problem

Although the procedure for stratified spin-up is essentially the same as for the homogeneous case, the mathematical analysis is much more complicated. In general, we cannot solve the complete problem at each order. But the essential lowest-order structure for the velocity, pressure and temperature can be obtained for time scales through the diffusion time. For the general case, we shall focus our attention on this part of the problem.

The special case $F = 1$ (F is the ratio of buoyancy frequency to rotational frequency) can be analysed in detail at each step. The reason for this is that the inertio-gravity waves generated are isotropic when $F = 1$, and the time and space dependence are separable. The results for this special case are inserted in the analysis since they serve as a partial check on the asymptotic treatment for arbitrary values of F.

Because of the greater complexity of the stratified problem we shall consider the simplest model. The basic state for $t < 0$ is solid-body rotation (Eddington–Sweet flow is neglected) and the fluid is stably stratified with a constant temperature gradient $4\Delta\overline{\Theta}/L$ (where $4\Delta\overline{\Theta}$ is the imposed temperature difference over the *half-depth* L). The fluid is Boussinesq. For $t > 0$ the upper and lower boundaries are moved with velocity $v = V_* \cos(kx_*/L)$. Since the forcing is independent of

y, the problem is two-dimensional. The magnitude of V_* is taken to be small (i.e. the problem is linear). We use the following definitions for dimensionless variables (asterisks refer to dimensional variables):

$$\mathbf{v}_* = V_* \mathbf{v}, \quad \mathbf{r}_* = L\mathbf{r}, \quad t = \Omega t_*, \theta_* = 2\Delta\Theta_* \theta,$$
$$p = [p_* - \rho_* \Omega^2 \tfrac{1}{2}(x_*^2 + y_*^2)]/\rho_* L V_* \Omega.$$

ρ_* is the constant reference density of the fluid. $2\Delta\Theta_*$ is the magnitude of the temperature difference generated by the velocity field. Finally, using the same three time scales as for the homogeneous problem, we write the non-dimensional equations in the form

$$U_t + E^{\frac{1}{2}} U_\tau + E U_T - 2V = -kP + E\nabla^2 U, \tag{5.1a}$$

$$V_t + E^{\frac{1}{2}} V_\tau + E V_T + 2U = E\nabla^2 V, \tag{5.1b}$$

$$W_t + E^{\frac{1}{2}} W_\tau + E W_T - 2F\Theta = -P_z + E\nabla^2 W, \tag{5.1c}$$

$$\Theta_t + E^{\frac{1}{2}} \Theta_\tau + E\Theta_T + 2FW = E/\sigma \nabla^2 \Theta, \tag{5.1d}$$

$$W_z = kU. \tag{5.1e}$$

The definitions of the variables are such that the factor 2 appears, as shown. The horizontal spatial dependence is taken care of by the relations

$$(u, v) = (U, V)\cos kx, \quad (w, p, \theta) = (W, P, \Theta)\sin kx,$$

so that all upper-case variables are independent of x. The non-dimensional parameters are: Prandtl number $\sigma = \nu/\kappa$ (κ is thermometric diffusivity); internal Froude number

$$F = N/\Omega = \left(\frac{-g}{\rho_*}\frac{\partial \rho_*}{\partial z_*}\right)^{\frac{1}{2}} \Big/ 2\Omega, \quad N^2 = g\alpha\Delta\overline{\Theta}/L;$$

Ekman number $E = \nu/\Omega L^2$. We assume $E^{\frac{1}{2}} \ll \sigma \ll E^{-\frac{1}{2}}$, $F \ll E^{-\frac{1}{2}}$, so that the system can be ordered with respect to $E^{\frac{1}{2}}$ only.

Because of the vertical symmetry of the problem, we consider the half-domain $0 \geqslant z \geqslant -1$. The boundary conditions are

$$\left.\begin{aligned} U = 0, \quad W = 0, \quad V = 1, \quad \text{at} \quad z = -1, \quad t > 0, \\ U_z = 0, \quad V_z = 0, \quad W = 0, \quad \text{at} \quad z = 0, \quad t > 0. \end{aligned}\right\} \tag{5.2}$$

If the boundary heat flux at $z = -1$ is fixed, we have

$$\Theta_z = 0 \quad \text{at} \quad z = -1, \quad t > 0. \tag{5.3}$$

If the boundary at $z = -1$ is kept at a fixed temperature,

$$\Theta = 0 \quad \text{at} \quad z = -1, \quad t > 0. \tag{5.4}$$

As in the homogeneous problem, the variables are divided into interior and boundary-layer parts, and each part is expanded in powers of $E^{\frac{1}{2}}$. Also, boundary values are indicated by superscript zero. The procedure is the same as before. We shall investigate the different orders of the interior equations, and use the boundary-layer results primarily to obtain conditions on the interior variables. Before undertaking the spin-up analysis, we shall look at the steady-state problem.

The steady problem

In the absence of transient processes, (5.1) reduce to

$$-2V = -kP + E\nabla^2 U, \quad 2U = E\nabla^2 V, \tag{5.5a, b}$$

$$-2F\Theta = -P_z + E\nabla^2 W, \quad 2FW = E/\sigma\nabla^2\Theta, \quad W_z = kU. \tag{5.5c–e}$$

The interior equations (subscript I) at lowest order reduce to the geostrophic-hydrostatic system

$$2V_I = kP_I, \quad P_{Iz} = 2F\Theta_I, \quad V_{Iz} = Fk\Theta_I. \tag{5.6}$$

Substituting (5.6) for V_I and Θ_I in (5.5b, d) and making use of (5.5e) yields

$$\nabla^2[P_{Izz} - \sigma F^2 k^2 P_I] = 0. \tag{5.7}$$

There is no Ekman pumping at orders E^0 or $E^{\frac{1}{2}}$. One can either go through the boundary-layer analysis to show this or one can observe that the interior vertical velocity is suppressed up to order E, so that there can be no Ekman pumping. Hence, the lowest-order boundary conditions (5.2) and (5.3) in terms of P_I are

$$V_I^0 = kP_I^0/2 = 1, \quad \Theta_{Iz}^0 = P_{Izz}^0/(2F) = 0; \tag{5.8}$$

the solution is

$$P_I = \frac{2}{k(\sigma F^2 - 1)}\left[\frac{\sigma F^2 \cosh kz}{\cosh k} - \frac{\cosh(\sigma^{\frac{1}{2}}Fkz)}{\cosh(\sigma^{\frac{1}{2}}Fk)}\right]. \tag{5.9}$$

When the boundary temperatures are fixed, (5.4) can be written as

$$\Theta_I^0 = P_{Iz}^0/(2F) = 0; \tag{5.10}$$

the solution is

$$P_I = \frac{2[\sinh k \cosh(\sigma^{\frac{1}{2}}Fkz) - \sigma^{\frac{1}{2}}F \sinh(\sigma^{\frac{1}{2}}Fkz)\cosh kz]}{k[\cosh(\sigma^{\frac{1}{2}}Fk)\sinh k - \sigma^{\frac{1}{2}}Fk \sinh(\sigma^{\frac{1}{2}}Fk)\cosh k]}. \tag{5.11}$$

We have derived these steady-state solutions to point out two features.

(i) Observe that for large k the response of the fluid is confined to a boundary layer the thickness of which is (crudely) k^{-1} if $1 < \sigma^{\frac{1}{2}}F$ or $(\sigma^{\frac{1}{2}}Fk)^{-1}$ if $1 > \sigma^{\frac{1}{2}}F$. The reason for this is that the system is controlled by diffusion, so that velocity variations of small scale imposed at the boundaries do not penetrate far into the fluid before being wiped out by horizontal diffusion. If k, σ and F are $O(1)$, the vertical variation of the response is smoother.

(ii) The final state to which the fluid tends in the transient problem is given by (5.9) or (5.11), depending on the thermal boundary condition. It is especially instructive to observe this z dependence because the usual stratified spin-up solution has a vertical structure determined by cosh kFz and it bears no resemblance to the steady-state solution. Therefore, in the stratified problem the usual spin-up solution is an intermediate stage through which the fluid must pass before approaching the spatial structure of the steady flow. We point this out here because the steady-state solution must emerge from the multi-timing analysis at an order higher than that corresponding to the usual spin-up problem. It turns out that this long-time behaviour comes in bits and pieces as the multi-timing analysis proceeds until finally a complete boundary-value problem is formulated. It helps to know *a priori* that that is going to happen.

Zero-order interior system

The lowest-order interior equations for the transient problem are

$$U_{0t}-2V_0 = -kP_0, \quad V_{0t}+2U_0 = 0, \quad W_{0t}-2F\Theta_0 = -P_{0z}, \qquad (5.12\,a\text{–}c)$$

$$W_{0z} = kU_0, \quad \Theta_{0t}+2FW_0 = 0. \qquad (5.12\,d,e)$$

These can be reduced to a single equation in W_0,

$$L_1W_{0zz}-k^2L_2W_0 = 0, \qquad (5.13)$$

where

$$L_1 \equiv \partial^2/\partial t^2+4, \quad L_2 \equiv \partial^2/\partial t^2+4F^2. \qquad (5.14)$$

As before, the lowest-order boundary-layer equations yield $\tilde{W}_0 = \tilde{P}_0 = 0$. Hence, since W vanishes at $z = 0, -1$, we note that W_0 vanishes at $z = 0, -1$; and we conclude from (5.13) that W_0 vanishes everywhere. The zero-order interior system then reduces to

$$\left.\begin{aligned} &U_0 = 0, \quad W_0 = 0, \quad V_0 = V_0(\tau,T,z),\\ &P_0 = P_0(\tau,T,z), \quad \Theta_0 = \Theta_0(\tau,T,z),\end{aligned}\right\} \qquad (5.15)$$

and

$$2V_0 = kP_0, \quad 2F\Theta_0 = P_{0z}, \quad V_{0z} = kF\Theta_0; \qquad (5.16)$$

i.e the flow is geostrophic and hydrostatic and satisfies the thermal-wind relation.

Zero-order boundary layers

The dynamical boundary-layer equations at zero order form the Ekman-layer system, the solution of which for this case is easily verified to be

$$\tilde{\phi}_0 = i(1-\tfrac{1}{2}kP_0^0)\int_0^t \frac{\zeta\exp(-\zeta^2/4\alpha)\exp(-2i\alpha)}{2\alpha(\pi\alpha)^{\frac{1}{2}}}\,d\alpha, \qquad (5.17)$$

where $\tilde{\phi}_0 = \tilde{U}_0+i\tilde{V}_0$. As in the homogeneous problem, we need to know $\tilde{W}_1^0$, which can be derived by integrating the continuity equation

$$\tilde{W}_{1\zeta} = k\tilde{U}_0 = k\,\mathrm{Re}\,\tilde{\phi}_0$$

from zero to infinity. We obtain

$$\tilde{W}_1^0 = -k(1-\tfrac{1}{2}kP_0^0)\,S(2t). \qquad (5.18)$$

The heat equation in the boundary layer reduces to

$$\tilde{\Theta}_{0t} = \sigma^{-1}\tilde{\Theta}_{0\zeta\zeta}, \qquad (5.19)$$

with solution

$$\tilde{\Theta}_0 = \tilde{\Theta}_0^0\,\mathrm{erfc}\,(\tfrac{1}{2}\zeta(t/\sigma)^{\frac{1}{2}}). \qquad (5.20)$$

For the fixed temperature case, (5.4) yields $\tilde{\Theta}_0^0 = -\Theta_0^0 = -P_{0z}^0/2F$, so (5.20) becomes

$$\tilde{\Theta}_0 = -P_{0z}^0/(2F)\,\mathrm{erfc}\,(\tfrac{1}{2}\zeta(t/\sigma)^{\frac{1}{2}}). \qquad (5.21)$$

When the heat flux is fixed, (5.3) yields $\tilde{\Theta}_{0\zeta}^0 = 0$, so (5.20) becomes

$$\tilde{\Theta}_0 = 0. \qquad (5.22)$$

First-order interior system

From (5.1) we obtain

$$U_{1t} - 2V_1 = -kP_1, \quad V_{1t} + 2U_1 = -V_{0\tau}, \quad W_{1t} - 2F\Theta_1 = -P_{1z}, \qquad (5.23\,a\text{–}c)$$

$$\Theta_{1t} + 2FW_1 = -\Theta_{0\tau}, \quad W_{1z} = kU_1, \qquad (5.23\,d, e)$$

where we have used the results $U_0 = 0$, $W_0 = 0$. In terms of P_1, this system can be expressed as

$$L_1 P_{1zzt} - L_2 k^2 P_{1t} = 4(k^2F^2P_{0\tau} - P_{0zz\tau}). \qquad (5.24)$$

Since P_0 is independent of t, we can integrate once with respect to t; the resulting equation has a forcing term on the right-hand side, proportional to t. Hence, in general, P_1 will grow linearly in t. But then the term $E^{\frac{1}{2}}P_1$ exceeds P_0 after a time $E^{-\frac{1}{2}}$, and the expansion is not uniformly valid. So we suppress the secular terms by requiring that

$$P_{0zz\tau} - k^2F^2P_{0\tau} = 0. \qquad (5.25)$$

This equation is the same as the one leading to the spin-up solution of Holton (1965).

We can integrate (5.25) with respect to τ, to obtain

$$P_{0zz} - k^2F^2P_0 = Q(T, z). \qquad (5.26)$$

$Q(T, z)$ is an arbitrary function of T and z, and vanishes initially.

To obtain a boundary condition for P_0, we evaluate (5.23d) at $z = -1$. Thus

$$\Theta^0_{1t} + 2FW^0_1 = -\Theta^0_{0\tau}, \qquad (5.27)$$

where $\Theta^0_{0\tau} = P^0_{0z\tau}/2F$ from (5.16). We use the non-secular condition on Θ^0_1, and require that $\Theta^0_{1t} \to 0$ as $t \to \infty$. Then, noting from (5.2) that $W^0_1 = -\tilde{W}^0_1$, we use (5.18) to obtain

$$W^0_1 \to \tfrac{1}{2}k\,(1 - \tfrac{1}{2}kP^0_0). \qquad (5.28)$$

Hence, as $t \to \infty$, (5.27) yields the asymptotic relation

$$P^0_{0z\tau} - k^2F^2P^0_0 = -2kF^2. \qquad (5.29)$$

The solution to the boundary-value problem given by (5.25), (5.26), and (5.29), with P_0 symmetric in z, includes the usual spin-up solution and can be written as

$$P_0 = \frac{2f(T)}{k}\,[1 - \exp(-\mu\tau)]\,\frac{\cosh(Fkz)}{\cosh(Fk)} + g(T, z,) \qquad (5.30\,a)$$

with

$$\mu = Fk\coth(Fk), \quad g(0, z) = 0, \quad f(0) = 1, \quad g^0(T, -1) = 2/k[1 - f(T)]. \quad (5.30\,b)$$

The term $(2f/k)\,[\cosh(Fkz)/\cosh(Fk)]$ can be absorbed into g but we give the result as shown, because with $f(T) = 1$ and $g = 0$, it has the form of the usual spin-up solution (as it must if we ignore T). Except for an initial and a boundary value, g is unspecified at this point. However, it is helpful to point out here that g will eventually contain the steady-state solution.

At the next order, we shall need the asymptotic value of W_1. From (5.23) we have

$$L_1 W_{1zz} - k^2 L_2 W_1 = 0, \tag{5.31}$$

and from (5.2) the boundary condition $W_1^0 = -\tilde{W}_1^0$ with $\tilde{W}_1^0$ given by (5.18). A Laplace transform in time and a solution to the transform problem lead to

$$\hat{W}_1 = -k(1-\tfrac{1}{2}kP_0^0)\,\widehat{S(2t)}\,\frac{\sinh kz\,[(s^2+4F^2)/(s^2+4)]^{\frac{1}{2}}}{\sinh k[(s^2+4F^2)/(s^2+4)]^{\frac{1}{2}}}. \tag{5.32}$$

We have not inverted the transform for arbitrary F. But when $F = 1$ the solution is

$$W_1 = -k(1-\tfrac{1}{2}kP_0^0)\,S(2t)\sinh(kz)/\sinh k. \tag{5.33}$$

For arbitrary F, if we obtain the asymptotic behaviour of W_1 as $t \to \infty$ from the limiting behaviour† of (5.32) as $s \to 0$, we get

$$W_1 \to -\tfrac{1}{2}(1-\tfrac{1}{2}kP_0^0)\sinh(kFz)/\sinh(kF). \tag{5.34}$$

First-order boundary layers

The dynamical boundary-layer equations at $O(E^{\frac{1}{2}})$ reduce to the Ekman-layer equation with two forcing terms

$$\tilde{\phi}_{1t} + 2i\tilde{\phi}_1 - \tilde{\phi}_{1\zeta\zeta} = -k\tilde{P}_1 - \tilde{\phi}_{0\tau}. \tag{5.35}$$

The term $\tilde{P}_1$ is evaluated from $\tilde{P}_{1\zeta} = 2F\tilde{\Theta}_0$ and $\tilde{\phi}_{0\tau}$ is obtained from (5.17). A Laplace transform in time and subsequent solution of the boundary-value problem in ζ lead to

$$\hat{\tilde{\phi}}_1 = \left[\hat{\tilde{\phi}}_1^0 - \frac{\hat{\tilde{\phi}}_{0\tau}\zeta}{2s\,(s+2i)^{\frac{1}{2}}}\right]\exp\{-(s+2i)^{\frac{1}{2}}\zeta\} - \frac{2kF[\Theta_0^0]}{s(\sigma s)^{\frac{1}{2}}\,(\sigma s - s - 2i)}$$
$$\times[\exp\{-(s+2i)^{\frac{1}{2}}\zeta\} - \exp\{-(\sigma s)^{\frac{1}{2}}\zeta\}]. \tag{5.36}$$

The last term comes from $\tilde{P}_1$ in (5.35) and $[\Theta_0^0] = \Theta_0^0$ for fixed boundary temperature and $[\Theta_0^0] = 0$ for fixed heat flux.

Again, our main use of $\tilde{\phi}_1$ will be to obtain a boundary value for W_2, which by (5.2) is $W_2^0 = -\tilde{W}_2^0$. We obtain $\tilde{W}_2^0$ by integrating the transform of the continuity equation $\tilde{W}_{2\zeta} = k\,\mathrm{Re}\,(\tilde{\phi}_1)$ from zero to infinity and making use of (5.36):

$$\hat{\tilde{W}}_2^0 = -\mathrm{Re}\left\{\frac{k\hat{\tilde{\phi}}_1^0}{(s+2i)^{\frac{1}{2}}} - \frac{k\hat{\tilde{\phi}}_{0\tau}^0}{2s(s+2i)^{\frac{3}{2}}} + \frac{2k^2F[\Theta_0^0]}{s(\sigma s)^{\frac{1}{2}}\,(\sigma s - s - 2i)}\,[(s+2i)^{-\frac{1}{2}} - (\sigma s)^{-\frac{1}{2}}]\right\}. \tag{5.37}$$

We obtain $k\hat{\tilde{\phi}}_1^0\ (= -k\hat{\phi}_1^0)$ from (5.23b), as

$$-k\hat{\tilde{\phi}}_1^0 = k\hat{\phi}_1^0 = \frac{s-2i}{s}\,\hat{W}_{1z}^0 - \frac{ik}{s^2}\,V_{0\tau}^0. \tag{5.38}$$

† We shall make liberal use of this Tauberian theorem. For $F = 1$ the result always agrees with the asymptotic behaviour derived from the exact solution, as it does here.

$\hat{W}^0_{1z}$ is easily evaluated from (5.32). Substituting (5.38) into (5.37) and expressing $\bar{\phi}^0_0$ in terms of P_0 yields

$$\hat{W}^0_2 = \mathrm{Re}\left\{\frac{-k^2(1-\frac{1}{2}kP^0_0)}{2is^2(s+2i)^{\frac{1}{2}}}(s-2i)\left(\frac{s^2+4F^2}{s^2+4}\right)^{\frac{1}{2}}\coth\left(k\left(\frac{s^2+4F^2}{s^2+4}\right)^{\frac{1}{2}}\right)\right.$$

$$\times[(s-2i)^{-\frac{1}{2}}-(s+2i)^{-\frac{1}{2}}]-\frac{ik^2P^0_{0\tau}}{2s(s+2i)^{\frac{1}{2}}}\left[\frac{1}{s}+\frac{1}{2(s+2i)}\right]$$

$$\left.+\frac{2k^2F[\Theta^0_0]}{s(\sigma s)^{\frac{1}{2}}(\sigma s-s-2i)}[(s+2i)^{-\frac{1}{2}}-(\sigma s)^{-\frac{1}{2}}]\right\}. \tag{5.39}$$

For $F = 1$, $\tilde{W}^0_2$ can be obtained directly:

$$\tilde{W}^0_2 = k^2(1-\tfrac{1}{2}kP^0_0)\coth k\left[\int_0^t(t-\alpha)J_0(2\alpha)\,d\alpha+\tfrac{1}{4}(\cos 2t-1)\right]$$

$$-\tfrac{1}{2}k^2P^0_{0\tau}tS(2t)+2k^2[\Theta^0_0]\left\{\frac{(\sigma-1)\cos[2t/(\sigma-1)]}{4\sigma}\right.$$

$$\left.+\frac{2}{\pi\sigma^{\frac{1}{2}}(\sigma-1)}\int_0^t(t-\beta)^{\frac{1}{2}}\int_0^\beta\frac{\exp\{2i[\alpha\sigma/(\sigma-1)-\beta]\}}{(\beta-\alpha)^{\frac{1}{2}}}\,d\alpha\,d\beta\right\}. \tag{5.40}$$

Second-order interior system

The interior equations at $O(E)$ are

$$U_{2t}-2V_2 = -kP_2-U_{1\tau}, \tag{5.41a}$$

$$V_{2t}+2U_2 = -V_{1\tau}-V_{0T}+\nabla^2V_0, \tag{5.41b}$$

$$W_{2t}-2F\Theta_2 = -P_{2z}-W_{1\tau}, \tag{5.41c}$$

$$\Theta_{2t}+2FW_2 = -\Theta_{1\tau}-\Theta_{0T}+\sigma^{-1}\nabla^2\Theta_0, \tag{5.41d}$$

$$W_{2z} = kU_2; \tag{5.41e}$$

or, in terms of P_2,

$$L_1P_{2zzt}-k^2L_2P_{2t} = 8(F^2-1)W_{1z\tau t}-4(P_{1zz\tau}-k^2F^2P_{1\tau})$$

$$-4(P_{0zzT}-k^2F^2P_{0T})+4\nabla^2(\sigma^{-1}P_{0zz}-k^2F^2P_0). \tag{5.42}$$

Once again we apply the non-secularity condition, namely, that the asymptotic value of the terms that involve derivatives with respect to short time t vanish. On the right-hand side, this condition is also satisfied by W_{izzt} (from (5.34)). The terms that remain must then balance, so we have, in the limit as $t\to\infty$,

$$P_{1zz\tau}-k^2F^2P_{1\tau}+P_{0zzT}-k^2F^2P_{0T}-\nabla^2(\sigma^{-1}P_{0zz}-k^2F^2P_0) = 0. \tag{5.43}$$

The analysis of (5.43) leads to the steady solution, (5.9) or (5.11), and to the long-time behaviour of P_0. To proceed we require boundary conditions to determine the part of P_0 that is still unknown, namely the forms of $f(T)$ and $g(T,z)$ in (5.30). Since (5.43) contains four derivatives in z, more boundary conditions are required than for the lower-order equations. It is at this point that the thermal boundary conditions appear effectively for the first time.

The remainder of this section is concerned with obtaining the solution to (5.43). The discussion is rather involved because some of the issues that arise are unfamiliar (at least, they were to us, when we first encountered them) and a detailed treatment is necessary to the argument. Our first task is to derive boundary conditions to go with (5.43).

Boundary conditions at second order

We have already obtained an expression for $\tilde{W}_2^0$ ($-W_2^0$). For the general case this boundary value is given by (5.39) in terms of the transformed variable $\hat{\tilde{W}}_2^0$; but for $F = 1$ it is given explicitly for $\tilde{W}_2^0$ by (5.40). For either expression it can be shown that, for large t,

$$\tilde{W}_2^0 \to [kF \coth kF\,(\tfrac{1}{2}k - \tfrac{1}{4}k^2P_0^0) - \tfrac{1}{4}k^2P_{0r}^0]\,t + \frac{k^2F[\Theta_0^0]\,t^{\frac{1}{2}}}{(\sigma\pi)^{\frac{1}{2}}} + A\left(\frac{t}{\pi}\right)^{\frac{1}{2}}\cos 2t + B. \quad (5.44)$$

A and B are non-growing terms involving P_0^0 and P_{0r}^0. From (5.39) the limit (5.44) is obtained by taking $s \to 0$ and then inverting. From (5.40) it can be derived directly.

In order that $\tilde{W}_2^0$ be non-secular it is necessary that the bracket multiplying t vanish. This condition is equivalent to (5.29). An additional condition emerges from (5.44). Even with the first bracket vanishing, the term $E\tilde{W}_2^0$ will grow to a value larger than $E^{\frac{1}{2}}\tilde{W}_1^0$ in a time $O(E^{-1})$, because of the term proportional to $t^{\frac{1}{2}}$. To keep our expansion valid for times of this order (i.e. for $T > 0$) we must require $[\Theta_0^0] = 0$. In the case of fixed heat flux, $[\Theta_0^0] \equiv 0$, so this term is not present in (5.44). However, when the boundary temperature is fixed, $[\Theta_0^0] \equiv \Theta_0^0$, so we require

$$\Theta_0^0 = P_{0z}^0/2F = 0 \quad \text{for} \quad T > 0. \quad (5.45)$$

$T > 0$ means that we impose this condition only for the problem associated with the long time scale.

We make two observations. (i) The condition $\Theta_0^0 = 0$ is identical to (5.10) for the steady problem with fixed boundary temperatures. (ii) This condition cannot be applied for times $O(1)$. If it were, the first-order problem would be overdetermined, as can be seen from (5.30a), where $\Theta_0^0 = P_{0z}^0/2F$ and $P_{0z}^0 \neq 0$.

When the heat flux is fixed, a different condition is obtained. For this case, $\tilde{\Theta}_0 \equiv 0$ by (5.22), and the first-order heat equation in the boundary layer can be expressed in terms of $\tilde{\Theta}_{1\zeta}$ as

$$(\tilde{\Theta}_{1\zeta})_{\zeta\zeta} = \sigma\,(\tilde{\Theta}_{1\zeta})_t + 2\sigma F\tilde{W}_{1\zeta}. \quad (5.46)$$

The boundary condition (5.3) to this order becomes

$$\tilde{\Theta}_{1\zeta}^0 = -\Theta_{0z}^0 = -P_{0zz}^0/2F. \quad (5.47)$$

The term $\tilde{W}_{1\zeta}$ is evaluated from the continuity equation and from (5.17). The solution for $\tilde{\Theta}_{1\zeta}$ is

$$\tilde{\Theta}_{1\zeta} = [kF\sigma(1 - \tfrac{1}{2}kP_0^0) - P_{0zz}^0/2F]\,\mathrm{erfc}\,(\tfrac{1}{2}\zeta(t/\sigma)^{\frac{1}{2}}) + \text{oscillatory terms.} \quad (5.48)$$

Now, when t is $O(1)$, the complementary error function vanishes as $\zeta \to \infty$. But, when t is $O(E^{-1})$, $\mathrm{erfc}\,(\zeta\sigma^{\frac{1}{2}}/2t^{\frac{1}{2}})$ gives an $O(1)$ contribution for values of ζ

up to $E^{-\frac{1}{2}}$. Thus the boundary-layer solution penetrates to the middle of the fluid layer. If we were to allow this to happen, our separation of Θ into interior and boundary layer parts would break down. Hence, to preserve our adopted procedure, we shall require that the term in brackets vanish, i.e.

$$k^2F^2\sigma(2/k-P_0^0)-P_{0zz}^0 = 0 \quad \text{for} \quad T > 0. \tag{5.49}$$

This condition is also applied only for long times. When we finally have the long-time problem formulated, it will be seen that (5.49) will reduce to the usual boundary condition for fixed heat flux for the steady part of the flow.

It is instructive to observe that (5.49) is required not just to preserve non-secular behaviour in time. It removes the penetration of the boundary layer into the interior, thus making the problem spatially non-secular. This aspect of the analysis is one that we have not encountered in any other problem; it is associated with the multi-scaling in time *and* space.

For the higher-order problem included in (5.43) we have now derived one boundary condition, given by (5.45) for fixed temperature and by (5.49) for fixed heat flux. The second condition is obtained from (5.41*d*) evaluated at the boundary. (This procedure is exactly analogous to the one for the first-order problem where we derived (5.29) from (5.27).) Thus,

$$\Theta_{2t}^0+\Theta_{1\tau}^0+\Theta_{0T}^0+2FW_2^0 = \sigma^{-1}\nabla^2\Theta_0^0. \tag{5.50}$$

Applying the non-secular condition to θ_2^0, we require

$$\lim_{t\to\infty} t^{-1}\int_0^t \Theta_{2t}^0 dt = 0. \tag{5.51}$$

There are several types of terms in (5.50). Those that would be secular themselves (i.e. the ones with t or $t^{\frac{1}{2}}$ as coefficients) have already been taken care of by the discussions following (5.27) and (5.44). Those that are proportional to $t^{\frac{1}{2}}\cos 2t$, or oscillate or decay, are non-secular in the sense of (5.51). The remaining terms are those that are asymptotically independent of t, and therefore do not satisfy (5.51). Obviously, since Θ_0^0 is independent of t by (5.15), *only* such terms appear in $\nabla^2\Theta_0^0$ and Θ_{0T}^0. Also, W_2^0 and $\Theta_{1\tau}^0$ contain parts that are independent of t. The total contribution of these terms must vanish for (5.51) to be valid, so we obtain a boundary condition by balancing those terms in (5.50).

To obtain the appropriate contribution for W_2^0 we could return to (5.44), since $\tilde{W}_2^0 = -W_2^0$, and determine the form of B. However, it is simpler to observe that we are seeking the non-vanishing, non-oscillatory asymptotic form of W_2^0. This can be obtained simply by taking the steady form of (5.35) (since $\lim_{t\to\infty}\phi_{1t}\to 0$), and evaluating $\tilde{W}_2^0$ as before. We derive

$$W_2^0 \to k^2(\tfrac{3}{16}P_{0\tau}^0-\tfrac{1}{4}P_1^0). \tag{5.52}$$

The terms in (5.50) that must balance in the limit as $t\to\infty$ reduce to

$$P_{1z\tau}^0-k^2F^2P_1^0 = -\tfrac{3}{4}k^2F^2P_{0\tau}^0-P_{0z\tau}^0+\sigma^{-1}\nabla^2P_{0z}^0. \tag{5.53}$$

To summarize the problem, therefore, we must solve (5.43) with the boundary condition (5.53), and either (5.45) or (5.49), depending on whether the boundary is at fixed temperature or fixed heat flux, respectively.

Solution

We first substitute the form (5.30*a*) into (5.43), to get

$$P_{1zz\tau}-k^2F^2P_{1\tau} = -g_{zzT}+k^2F^2g_T+\sigma^{-1}g_{zzzz}-k^2(\sigma^{-1}+F^2)\,g_{zz}+k^4F^2g + Af(T)\frac{\cosh(kFz)}{\cosh(kF)}(1-\exp(-\mu\tau)), \qquad (5.54)$$

where $A = 2k^3F^2(F^2-1)(\sigma^{-1}-1)$.

Now observe that all the terms on the right-hand side, except for the last, are independent of τ. Hence, these must balance, because otherwise P_1 would be secular in τ. Therefore, (5.54) turns into two equations: the first

$$-g_{zzT}+k^2F^2g_T+\sigma^{-1}g_{zzzz}-k^2(\sigma^{-1}+F^2)\,g_{zz}+k^4F^2g = -Af\frac{\cosh(kFz)}{\cosh(kF)}, \qquad (5.55)$$

and the second

$$P_{1zz\tau}-k^2F^2P_{1\tau} = -Af\frac{\cosh(kFz)}{\cosh(kF)}\exp(-\mu\tau). \qquad (5.56)$$

Because of the symmetry of P_1 in z, the solution to (5.56) has the form

$$P_{1\tau} = C(\tau,T)\frac{\cosh(kFz)}{\cosh(kF)}-\frac{fA\exp(-\mu\tau)}{2kF}\,\frac{z\sinh(kFz)}{\cosh(kF)}. \qquad (5.57)$$

$C(\tau, T)$ is as yet undetermined. The boundary condition (5.53) also separates into two equations, one of which includes the τ-dependent part of P_0 from (5.30*a*). Substituting (5.57) into the τ derivative of (5.53) yields

$$kF\tanh kFC_\tau+k^2F^2C = -\left\{\frac{Af\mu}{2kF}(\tanh(kf)+kF)-\frac{AfkF}{2}\tanh(kF) + \frac{2}{k}\left[\tfrac{3}{4}F^2k^2\mu^2f+\mu kF\tanh(kF)f_T-\frac{\mu k^3F}{\sigma}(F^2-1)\tanh(kF)f\right]\right\}\exp(-\mu\tau). \qquad (5.58)$$

This equation is the second-order analogue of the first-order equation that yielded the τ dependence of $\exp(-\mu\tau)$. In the present case we note that $\exp(-\mu\tau)$ on the right-hand side has the form of the solution to the differential operator on the left-hand side. Hence, this equation will yield a secular term of the form $\tau\exp(-\mu\tau)$. To suppress this secular term, we require that the right-hand side vanish. Thus, we finally obtain a differential equation for $f(T)$. After some manipulation we find

$$f_T+\beta f_0 = 0, \qquad (5.59)$$

where

$$\beta = \tfrac{3}{4}\mu^2-k^2(F^2-1)\left[\tfrac{1}{2}(\sigma^{-1}+1)-(\sigma^{-1}-1)\frac{kF}{\sinh(2kF)}\right]. \qquad (5.60)$$

The solution satisfying the initial condition (5.30*b*) (i.e. $f(0) = 1$) is

$$f = \exp(-\beta T). \qquad (5.61)$$

Hence, the solution to the elementary spin-up part of the problem is finally complete.

Note that, when F vanishes, $\beta = \frac{3}{4}+k^2$. This is the form of the equivalent term for the homogeneous problem with harmonic forcing and it serves as a (mild) check on our solution.

In the course of our development we have formulated the problem for P_1 as a natural step in the ordered system of equations. The evaluation of $f(T)$ is a consequence of applying the non-secular condition to P_1. We shall not continue with the solution to the problem for P_1, since this is a higher-order quantity. Our goal has been to complete the solution for P_0, so we now turn to the determination of the remaining part g.

The system to obtain g is made up of (5.55), the initial and one boundary condition in (5.30*b*) and either (5.45) or (5.49). It is convenient to divide g into a homogeneous part h and a particular part g_p, which takes care of the forcing term in (5.55). Thus, let g be written as

$$g = g_p + h(T, z), \tag{5.62a}$$

where

$$g_p = -\frac{2}{k}\frac{\cosh(kFz)}{\cosh(kF)}\exp(-\beta T). \tag{5.62b}$$

Then conditions (5.30*b*) can be expressed in terms of h as

$$h(0,z) = \frac{2}{k}\frac{\cosh(kFz)}{\cosh(kF)}, \quad h(T,-1) = \frac{2}{k}, \tag{5.63}$$

where again we have made use of vertical symmetry. The partial differential equation (5.55) can then be written in terms of h:

$$h_{zzT} - k^2F^2h_T = \sigma^{-1}h_{zzzz} - k^2(F^2+\sigma^{-1})\,h_{zz} + k^4F^2h. \tag{5.64}$$

And finally the conditions (5.45) and (5.49) become

$$h_z(T,-1) = 0, \quad \text{fixed temperature,} \tag{5.65a}$$

$$h_{zz}(T,-1) = 0, \quad \text{fixed heat flux.} \tag{5.65b}$$

The problems formulated here for h are similar to those posed by Pedlosky (1967), Allen (1973) and Clark (1973), although in our case $\sigma \gg E^{\frac{1}{2}}$, whereas Clark treats $\sigma = O(E^{\frac{1}{2}})$. We concentrate here on the case with fixed-heat-flux boundaries (condition 5.65*b*). Since the boundary conditions (5.63) and (5.65*b*) are independent of time, they must be satisfied by a time-independent part of h, which we denote by h_s. Thus

$$h_s = \frac{2}{k(1-\sigma F^2)}\left[\frac{\cosh\sigma^{\frac{1}{2}}kFz}{\cosh\sigma^{\frac{1}{2}}kF} - \sigma F^2\frac{\cosh kz}{\cosh k}\right]; \tag{5.66}$$

and we observe that h_s is identical to the steady-state solution (5.9). We then write h as

$$h = h_s + \sum_{n=0}^{\infty} A_n\cos\alpha_n z, \quad \alpha_n = (n+\tfrac{1}{2})\pi, \tag{5.67}$$

and substitute it into (5.64), to obtain

$$A_n = A_n^0\exp\left\{-\frac{(\alpha_n^2+k^2)(\alpha_n^2+\sigma k^2F^2)}{\sigma(\alpha_n^2+k^2F^2)}T\right\}. \tag{5.68}$$

The initial condition (5.63) is satisfied by

$$A_n^0 = (-1)^n\frac{4\alpha_n}{k}\left\{\frac{1}{\alpha_n^2+k^2F^2} - \frac{1}{1-\sigma F^2}\left[\frac{1}{\alpha_n^2+\sigma k^2F^2} - \frac{\sigma F^2}{\alpha_n^2+k^2}\right]\right\}. \tag{5.69}$$

Hence, the solution to the lowest-order problem is complete and we have

$$P_0 = -\frac{2}{k}\frac{\cosh(kFz)}{\cosh(kF)}\exp(-\mu\tau-\beta T)+h, \tag{5.70}$$

where

$$\mu = kF\coth(kF), \quad \beta = \tfrac{3}{4}\mu^2 - k^2(F^2-1)\,[\tfrac{1}{2}(\sigma+1)-(\sigma^{-1}-1)\,kF/\sinh(2kF)]$$

and h is given by (5.66)–(5.69).

6. Results for stratified solution

We have calculated results for two cases to show how the system behaves as a function of time and space. The values of the parameters for the two cases are shown in table 1. The Ekman and Prandtl numbers are typical for laboratory experiments with thermally stratified water. The two values of F exhibit the behaviour when rotation is the more important constraint ($F = 0{\cdot}5$) and when stratification is more important ($F = 1{\cdot}5$). With $k = 2$ the graphs for P_0 and V_0 are identical, since $V_0 = \frac{1}{2}kP_0$. The (dimensionless) homogeneous spin-up time for both cases is $\tau_h = E^{\frac{1}{2}} = 100$. The stratified spin-up time is $\tau_s = \tau_h/kF\coth(kF)$ and is given in units of τ_h in table 1.

In figure 1 two sets of graphs for case 1 are shown for V_0 as a function of z. One set shows the complete zero-order interior solution V_0 for different values of time (which we have taken in units of τ_s). Since an Ekman boundary layer is also present for shorter times, V_0 by itself does not satisfy the boundary conditions until the boundary-layer flow is dissipated. The latter occurs by about five stratified spin-up times (longer for smaller σ); and at this stage the flow has nearly achieved the steady-state form.

Observe that initially the amplitude of V_0 approaches the steady-state solution from below, as it must, since initially $V_0 \equiv 0$. At about $\tau = 2\tau_s$, the value of V_0 at $z = 0$ reaches the steady-state value, then it overshoots it, and finally approaches the steady value from above as time increases. This overshoot occurs for all values of z, but the closer to the boundary the later it occurs and the smaller the overshoot. For $F < 1$, $\sigma > 1$, the simple spin-up solution penetrates farther into the interior than does the steady-state solution (as can be seen from (5.66) and (5.70)). Hence, the overshoot is associated with the simple spin-up solution. The amplitude of the overshoot is only 4% at $z = 0$.

The second set of graphs in figure 1 shows how the long-time part of V_0 (i.e. the part corresponding to h in (5.70)) approaches the steady-state solution. The 'initial' distribution for h is equal to the distribution given by the elementary spin-up solution as $\tau \to \infty$ (i.e. $\cosh(kFz)/\cosh(kF)$). Hence, h approaches h_s from above. Also, since h corresponds to the long-time behaviour, there is no Ekman layer for this case, so that h satisfies the boundary conditions at all times.

Figure 2 shows the same two sets of graphs for case 2. Since $F > 1$, the larger effect of stratification confines the elementary spin-up response to a smaller depth than that of the steady response, so there is no overshoot. At earlier times, however, the spin-up layer shows up as a boundary layer. Also, for this value of σ, the steady state is essentially achieved after about 3 spin-up times.

	E	σ	K	F	τ_s/τ_h
Case 1	10^{-4}	7	2	0·5	0·77
Case 2	10^{-4}	7	2	1·5	0·33

TABLE 1. Values of parameters for calculated results

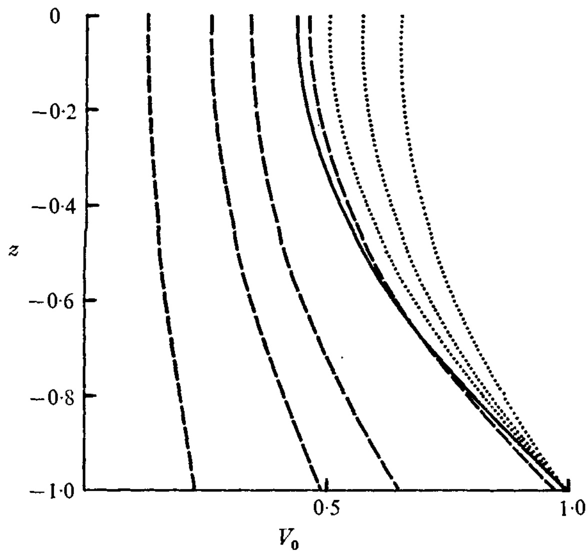

FIGURE 1. The zonal interior velocity V_0, shown as a function of z and time for case 1 of table 1 ($F = 0{\cdot}5$). From *left* to *right* the dashed curves correspond to times $\tau/\tau_s = 0{\cdot}25, 0{\cdot}65, 1{\cdot}0, 3{\cdot}5$. The solid curve is the steady-state solution. The complete spin-up solution approaches the steady solution from below, but overshoots it near the centre, and eventually settles down to the steady state. The dotted curves correspond to h, the long-time part of the solution, at times $\tau/\tau_s = 0, 1{\cdot}0, 2{\cdot}5$, from *right* to *left*. This part of the solution approaches the steady state from above. Complete spin-up is effectively achieved when $\tau = 5\tau_s$.

In recent experimental studies, Buzyna & Veronis (1971) and Saunders & Beardsley (1974, private communication) have found that experiment and elementary spin-up theory show consistent disagreements. A. Barcilon, J. Lau, S. Piacsek & A. Warn-Varnar (1974, private communication) have suggested that the discrepancy may be associated with the fact that the short-time behaviour has been neglected in the theoretical solutions.

Even though our study includes only a single wavenumber, the results may shed some light on the observed discrepancy. For an experimental set-up with a roughly square cross-sectional geometry, the gravest Fourier or Bessel mode of the boundary velocity has the largest effect on the interior flow. The value of $k = 2$ corresponds to the gravest mode for a cross-sectional geometry where the horizontal scale is $\frac{1}{2}\pi$ times the half-depth. Apart from effects due to cylindrical spreading, this is approximately the geometry of the experiments cited above. Hence, we may expect that the theory with $k = 2$ may give qualitatively correct information near $z = 0$, where the main effect is due to the gravest mode.

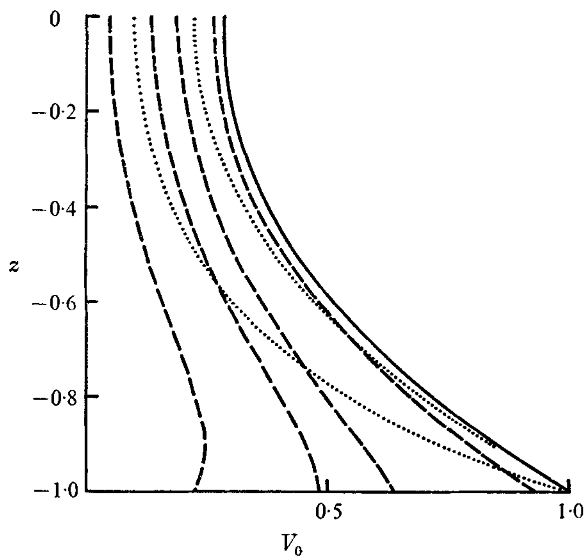

FIGURE 2. The zonal interior velocity V_0 shown as a function of z and of time for case 2 of table 1 ($F = 1\cdot5$). From *left* to *right* the dashed curves correspond to times $\tau/\tau_s = 0\cdot25, 0\cdot65, 1\cdot0, 2\cdot5$. The solid curve is the steady-state solution. With $F > 1$, the complete spin-up solution approaches the steady solution monotonically with time (no overshoot). The dotted curves correspond to h, the long-time part of the solution, at times $\tau/\tau_s = 0, 1\cdot0$, from *left* to *right* so h approaches the steady solution from below.

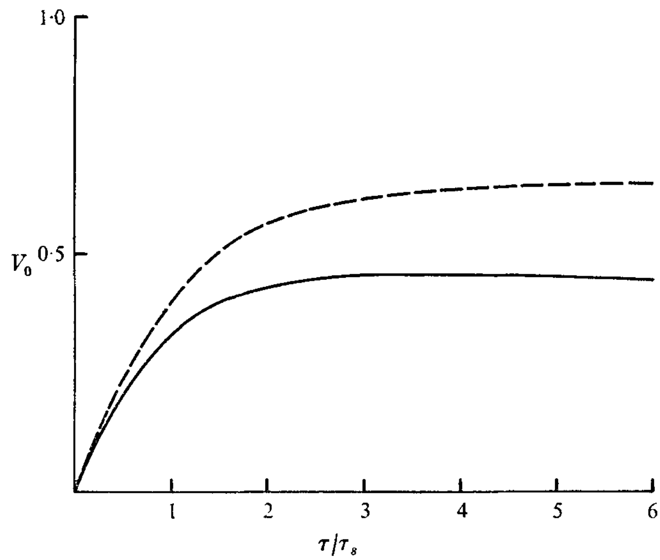

FIGURE 3. The elementary spin-up solution (dashed curve) and the complete lowest-order spin-up solution (solid curve) for V_0 at $z = 0$ as functions of time for $F = 0\cdot5$. The elementary solution over-estimates the response.

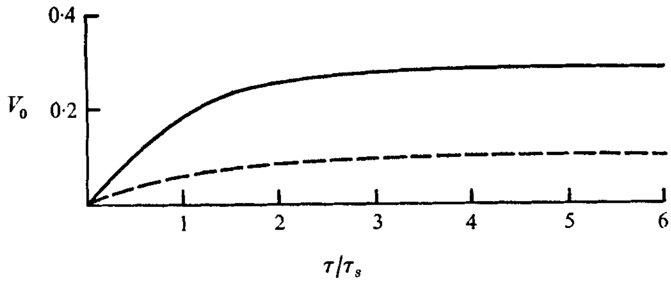

FIGURE 4. The elementary spin up solution (dashed curve) and the complete lowest-order spin-up solution (solid curve) for V_0 at $z = 0$ shown as functions of time for $F = 1{\cdot}5$. The elementary solution under-estimates the response.

Figure 3 compares the time distribution of the complete response V_0 for case 1 with that predicted by the simple spin-up theory. The elementary spin-up solution shows a more rapid growth than that of the complete solution. Thus, it over-estimates the spin-up response. Figure 4 shows that, for $F > 1$, the simple spin-up solution is an under-estimate of the complete response. The same qualitative discrepancy is present when experiments are compared with the simple solutions. Hence, it appears from our study that the observed discrepancy is at least partly due to the fact that the simple solution is not quantitatively correct, even for times of the order of one spin-up time.

We have presented results only for a single value of σ and for a single horizontal wavenumber. For larger k, the response is more confined to the boundary region and the spin-up time is shorter (Walin 1969). Hence, if the imposed boundary velocity is not a simple harmonic, but is decomposed into a Fourier series of harmonics, the gravest mode will penetrate farthest into the interior (see Buzyna & Veronis (1971) for a discussion of such a case). Also, the vertical structure of the steady solution is partly determined by the combination $\sigma F^{\frac{1}{2}}$. Hence, even for large F, sufficiently small σ can give rise to deep penetration. The physical reason for this is that small σ corresponds to large diffusion of heat, and the latter can serve to short circuit the effects of stratification. (Sakurai, Clark & Clark (1971) discussed this.) A third point is that, if the system has a relatively square cross-sectional geometry, the Fourier decomposition (Bessel functions for a cylinder) will always lead to structure in the vertical. For an actual experiment the final, solid-body rotation is achieved by lateral diffusion of properties from the side walls.

7. Summary and discussion of the stratified problem

The procedure that we have used involves interior and boundary-layer sets of equations at each order. At $O(E^0)$ we obtain the hydrostatic–geostrophic system and a transient Ekman-layer solution with coefficients expressed in terms of the interior variables and functions of the longer-time variables τ and T. The interior equations at $O(E^{\frac{1}{2}})$ include explicit time behaviour on two time scales, t and τ. By evaluating the boundary values of the variables with the help of those of the boundary-layer system, taking the limit as $t \to \infty$ and suppressing secular growth, we obtain the explicit dependence of the zero-order variables on τ. This is the usual spin-up solution if long time is neglected. At this stage the final steady solution appears as an undetermined function. The interior equations at $O(E)$ involve all three times, t, τ and T. Again combining interior and boundary-layer variables at the boundary, we end up with an expression for the vertical velocity that involves all three time scales. It is at this stage also that the thermal boundary conditions enter explicitly. Suppressing secular growth in time yields a boundary condition for the long-time problem that is equivalent to the boundary condition for the steady problem when the boundary temperature is fixed. For the fixed-heat-flux problem, the proper boundary condition emerges when we require that the boundary layer not penetrate to the interior. We interpret this as a non-secular condition in space. A second boundary condition, obtained by making use of Ekman suction at this order, is evaluated for large values of t and when secular growth on the τ time scale is suppressed, the long-time correction to the ordinary spin-up solution is obtained. Having taken care of these special properties, we are left with a boundary-value problem which includes the long-time approach to the final steady-state solution.

At $O(E^{\frac{1}{2}})$ in the above procedure the problem for the first-order variables first appears. We have not carried out the analysis beyond the zero-order variables. However, it should be observed that, just as in the homogeneous problem, the lowest-order solution for short times $\tau \ll 1$ is $O(E^{\frac{1}{2}})$, and the first-order correction is of the same order. It is only for large values of t (or for $\tau \sim 1$) that our solution is quantitatively dominant.

At the end of §4 we mentioned that, in the homogeneous problem, one can ignore the t time scale and still derive the results for the τ and T scales. For the stratified case that procedure runs into difficulties, because the zero-order boundary-layer equation for temperature reduces to $\tilde{\Theta}_{0\zeta\zeta} = 0$, so the solution is $\tilde{\Theta}_0 = 0$. Thus, for fixed temperature boundaries the condition $\Theta_0^0 = 0$ must be satisfied from the outset; and, as we mentioned earlier, this leads to an over-determined problem for P_0.

According to our analysis, the thermal boundary conditions should be applied only for the long-time problem. When the boundary temperature is fixed, the time at which the boundary condition is imposed follows from the non-secular requirement for W_2^0. When the heat flux is fixed, the argument is based on the separation of the variables into boundary-layer and interior parts. To preserve this separation for large times we require that the boundary-layer part not penetrate into the interior, and in this sense the system must be spatially non-secular.

Having carried out the foregoing analysis, one can do the simpler analysis for only the τ and T time scales, imposing the thermal conditions at the boundaries only for the long-time problem. This procedure would appear to be arbitrary without the justification given here.

Equation (5.57) contains the term $z \sinh(kFz)$, which looks secular in space. It would seem that the problem could be formulated in such a way as to remove this type of secularity also. We have not attempted to do so here, partly because of the secularity is associated with kF rather than $E^{\frac{1}{2}}$, and therefore is not affected by our ordering scheme, and partly because it has to do with completing the solution for P_1, which is not essential for our lowest-order result.

The complete solution for P_0 contains the term $\cosh kFz \exp(-\mu\tau - \beta T)$, where the factor $\exp(-\beta T)$ is the long-time correction. For moderately large values of kF, the exponential time dependence becomes

$$-\mu\tau - \beta T \rightarrow -kFE^{\frac{1}{2}}t\left[1+\left(\frac{1}{4}-\frac{1}{2\sigma}\right)kFE^{\frac{1}{2}}\right].$$

Hence, we see that the expansion scheme is valid provided that $kFE^{\frac{1}{2}} < 1$ or $kF < E^{-\frac{1}{2}}$ (i.e. the thickness of the elementary spin-up layer must exceed the thickness of the Ekman layer). This restriction is obviously necessary for our approach, in which the Ekman layer is treated as 'thin.'

For astrophysical applications, the limits $kF \gg 1$ and $\sigma \ll 1$ are important. For this range the time dependence becomes

$$-\mu\tau - \beta T \rightarrow -kFE^{\frac{1}{2}}t(1-(kF/2\sigma)E^{\frac{1}{2}}).$$

The condition for convergence is now $kF/\sigma < 2E^{-\frac{1}{2}}$, so that the restrictions on the values of k, F and σ are more stringent than we assumed initially. We note, too, that in this limit the elementary spin-up layer is spun up more slowly, because of the long-time correction (β is negative). However, complete spin-up is achieved more rapidly. Thus, the elementary spin-up process is not so pertinent in this case, even though it lasts longer.

Sakurai *et al.* (1971) pointed out the following. (i) For $\sigma > O(E^{\frac{1}{2}})$ the system is much like the one with $\sigma = O(E^0)$ (although for the validity of the simple spin-up solution it is necessary that $kF/\sigma < E^{-\frac{1}{2}}$). (ii) For $\sigma < O(E^{\frac{1}{2}})$ thermal diffusion is so strong that thermal effects are short-circuited and the fluid spins up as if it were homogeneous. The case with $\sigma = O(E^{\frac{1}{2}})$ is special, since the diffusion term becomes $O(E^{\frac{1}{2}})$, so Ekman suction is modified by diffusion. For this case, it is easy to show that the lowest-order closed system reduces to

$$(1/\sigma^*)\nabla^2 P_{0zz} - P_{0zz\tau} + k^2F^2P_{0\tau} = 0,$$

where $\sigma = \sigma^* E^{\frac{1}{2}}$, $\sigma^* = O(E^0)$, $\tau = E^{\frac{1}{2}}t$. If spatial derivatives are $O(E^0)$, the characteristic time is $O(\sigma^* F^2k^2E^{-\frac{1}{2}}\Omega^{-1})$ or $O(\sigma F^2k^2E^{-1}\Omega^{-1})$. This time is essentially the Eddington–Sweet time (Howard, Moore & Spiegel 1967); it is used as the basic time in Clark's (1973) treatment. For this problem with fixed temperature boundaries, the boundary condition on the interior flow obtained from the Ekman layer is

$$(1/\sigma^*)P^0_{0zzz} + k^2F^2P^0_0 = 2kF^2.$$

Hence, time appears in this analysis only in the form of the Eddington–Sweet time. Elementary spin-up time *per se* does not enter at all; nor does the spatial structure of the simple spin-up solution. Clark (1973) treated this problem in some detail; the interested reader is referred to his paper for the analysis. We cannot compare our detailed results with his because the case $\sigma = O(E^{\frac{1}{2}})$ must be analysed *ab initio* as special. The exact solution for the latter problem can be derived by the same procedure we have used to derive h; but the eigenfunctions for the infinite series are more complicated than the simple trigonometric functions we used in (5.67).

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